ZPolyTrans: a library for computing and enumerating integer transformations of Z-Polyhedra

Rachid Seghir

(Vincent Loechner because the french embassy did not deliver a visa to him)

Department of Computer Science
University of Batna, Algeria
seghir@univ-batna.dz

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What is ZPolyTrans?

- **ZPolyTrans** is a program written in C for:
  - computing integer affine transformations of parametric \( \mathbb{Z} \)-Polytopes,
  - counting integer points in the result of such transformations (unions of parametric \( \mathbb{Z} \)-polytopes).

- Based on *PolyLib*: for doing polyhedral and matrix operations.

- Based on *Barvinok* library: for counting integer points in a parametric polytope.

- Available on [http://zpolytrans.gforge.inria.fr](http://zpolytrans.gforge.inria.fr)
Motivation

Code transformation, optimization and parallelization techniques → computing and enumerating the integer affine transformations of parametric $\mathbb{Z}$-polyhedra.

- Array linearization for hardware design [Turjan et al. 2005],
- Memory size computation [Zhao and Malik 2000, Zhu et al. 2006],
- Data distribution for NUMA-machines [Heine and Slowik 2000],
- ...
Example

Consider two loop nests accessing an array \( A \).

```plaintext
for i = 0 to n do 
    for j = i+1 to n do 
        A[4*i+2*j] = ...

for k = 0 to n do 
    for l = 0 to n do 
        A[k+l] = ...
```

Question: which array elements are accessed?

\textit{ZPolyTrans} computes:

- the \textbf{accessed array elements} as a union of three \( \mathbb{Z} \)-polytopes:
  \[
  \{2 \leq x \leq 6n - 8 \land x \text{ even}\} \cup \{x = 6n - 4 \land n \geq 1 \land x \text{ even}\} \cup \{0 \leq x \leq 2n \land x \in \mathbb{Z}\},
  \]

- the \textbf{number of accessed elements} as a piecewise quasi-polynomial, equal to 1 when \( n = 0 \), 3 when \( n = 1 \) and \( 4n - 2 \) when \( n \geq 2 \).
Let $\mathcal{Z} = P_p \cap L$ be a parametric $\mathbb{Z}$-polytope:

$P_p = \{ x \in \mathbb{Q}^d \mid Ax \geq Bp + c \}$ a parametric rational polytope,

$L = \{ A'z + c' \mid z \in \mathbb{Z}^d \}$ a lattice.

An affine function $T : \mathbb{Z}^d \rightarrow \mathbb{Z}^k$

$x \mapsto x' = A''x + c''$

The transformation of $\mathcal{Z}$ by $T$ is a Presburger formula

$T(\mathcal{Z}) = \{ x' \in \mathbb{Z}^k \mid \exists x, z \in \mathbb{Z}^d,$

$Ax \geq Bp + c, x = A'z + c', x' = A''x + c'' \}$
The elimination of the existential variables is done in two steps:

1. Some existential variables can be eliminated by removing equalities implying them (the result is \(\mathbb{Z}\)-polytope).

2. Recursively eliminating the remaining existential variables from the \(\mathbb{Z}\)-polytope produced at the first step.

   - rewrite it as a set of lower and upper bounds on the variable to be eliminated \(l(x) \leq \beta z, \alpha z \leq u(x)\)
   - the result is given by the intersection of the integer projections of all its pairs of lower and upper bounds.
The projection of a pair of bounds is given in the form:

dark shadow $\cup \{\text{exact shadow} \cap \text{sub-lattices of hyperplanes}\}$

Projection of the pair of bounds $\{x + 2 \leq 3y, \ 2y \leq x + 1\}$
Example of the existential variables elimination

- Elimination of the existential variables from
  \[ P = \{ x \in \mathbb{Z} \mid \exists (i,j,k) \in \mathbb{Z}^3 : 1 \leq i \leq p + 4 \land \\
  1 \leq j \leq 5 \land 3 \leq 3k \leq 2p \land x = 3i + 4j + 6k + 1 \} \]

- After removing the equality
  \[
  \begin{cases}
    x \in \mathbb{Z} & \exists (i',k) \in \mathbb{Z}^2 : -3x - 2k \leq 4i' \leq -3x - 2k + p + 3 \land \\
    3 \leq 3k \leq 2p & -2x - 3k - 6 \leq 3i' \leq -2x - 3k - 2 \land
  \end{cases}
  \]

- Result of the rational elimination of \( i' \) (when \( p = 4 \)).
Example of the existential variables elimination (2)

- Result of the **integer** elimination of $i'$ (when $p = 4$).

- Result of the **integer** elimination of $k$ (when $p = 4$).

- Number of integer points in the projection (for $p \geq 2$)
  - Integer projection: $\mathcal{E}(p) = 7p + [14, 10, 12]_p$
  - $\neq$ rational projection: $\mathcal{E}(p) = 7p + 20$
The integer projection of a parametric $\mathbb{Z}$-polytope may result in a non-disjoint union of parametric $\mathbb{Z}$-polytopes.

To compute the number of integer points in such a union,

- apply the inclusion-exclusion principle to the original set (rather than computing its disjoint union)
- call to $Barvinok$ library to calculate the Ehrhart quasi-polynomial corresponding to each $\mathbb{Z}$-polytope and their intersections (if non empty).

The results are finally combined (addition/substraction) into a single quasi-polynomial.
Many approaches have been proposed:

- The work of W. Pugh [Pugh, 1994] on integer affine projections based on the Fourier-Motzkin variable elimination.

- The theoretical rational generating function based approach [Barvinok and Woods, 2002]; [Verdoolaege and Woods, 2005]; [Koeppe et al., 2008].

- The weak quantifier elimination approach [Lasaruk and Sturm 2007].

- The \( \mathbb{Z} \)-polyhedral model [Gautam and Rajopadhye, 2007]; [Iooss and Rajopadhye, 2012].

- The work of [Verdoolaege et al. 2005] based on applying rewriting rules and solving a parametric integer linear programming problem, implemented in Barvinok library.
To the demo!