Integer Set Coalescing

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INRIA and KU Leuven

January 19, 2015

Outline

1. Introduction and Motivation
   - Polyhedal Model
   - The need for coalescing
   - Traditional “Coalescing”
2. Coalescing in `isl`
   - Rational Cases
   - Constraints adjacent to inequality
   - Constraints adjacent to equality
   - Wrapping
   - Existentially Quantified Variables
3. Conclusions

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Polyhedral Model

```
R:   h(A[2]);
     for (int i = 0; i < 2; ++i)
         for (int j = 0; j < 2; ++j)
S:               A[i + j] = f(i, j);
     for (int k = 0; k < 2; ++k)
T:               g(A[k], A[0]);
```

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- Instance set (set of statement instances)

$$I = \{ \mathtt{R}(); \mathtt{S}(0,0); \mathtt{S}(0,1); \mathtt{S}(1,0); \mathtt{S}(1,1); \mathtt{T}(0); \mathtt{T}(1) \}$$

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$$= \{\, \mathtt{R}(); \mathtt{S}(i,j) : 0 \leq i < 2 \wedge 0 \leq j < 2; \mathtt{T}(k) : 0 \leq k < 2 \,\}$$

Equivalent Representations

extensive $\{ S(0,0); S(0,1); S(1,0); S(1,1) \}$

$$= \{ S(i,j) : (i = 0 \wedge j = 0) \vee (i = 0 \wedge j = 1) \vee (i = 1 \wedge j = 0) \vee (i = 1 \wedge j = 1) \}$$

intensive $\{ S(i,j) : 0 \leq i < 2 \wedge 0 \leq j < 2 \}$

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In general, representation with fewer disjuncts is preferred

- (usually) occupies less memory
- operations can be performed more efficiently
- the outcome of some operations depends on chosen representation
  - transitive closure approximation
  - AST generation

$\Rightarrow$ coalescing: replace representation by one with fewer disjuncts

Effect on AST Generation — guide

Without coalescing input

$$\{\,\mathtt{S1}(i) \to (i) : (1 \le i \le N \wedge i \le 2M) \vee (1 \le i \le N \wedge i \ge M);$$
$$\mathtt{S2}(i) \to (i) : (N + 1 \le i \le 2N)\,\}$$

```
for (int c0 = 1; c0 <= min(2 * M, N); c0 += 1)
  S1(c0);
for (int c0 = max(1, 2 * M + 1); c0 <= N; c0 += 1)
  S1(c0);
for (int c0 = N + 1; c0 <= 2 * N; c0 += 1)
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```

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After coalescing input

$$\{\, \mathtt{S1}(i) \to (i) : 1 \le i \le N; \mathtt{S2}(i) \to (i) : (N + 1 \le i \le 2N) \,\}$$

```
for (int c0 = 1; c0 <= N; c0 += 1)
  S1(c0);
for (int c0 = N + 1; c0 <= 2 * N; c0 += 1)
  S2(c0);
```

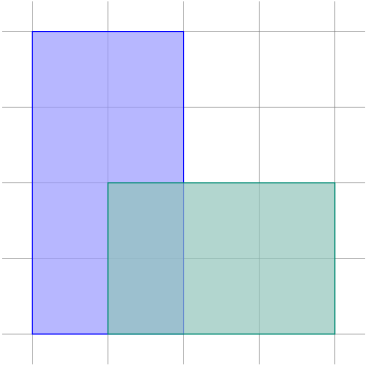
Effect on AST Generation — `cholesky`

⇒ demo

Causes of Splintering

Several operations on integer sets may introduce coalescing opportunities

- Projection



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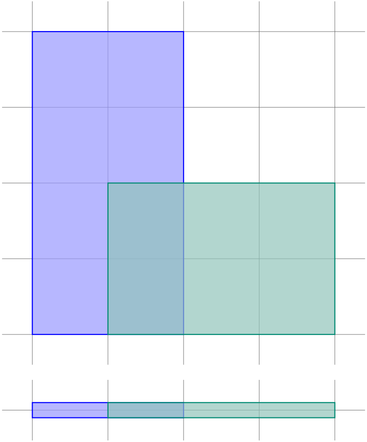

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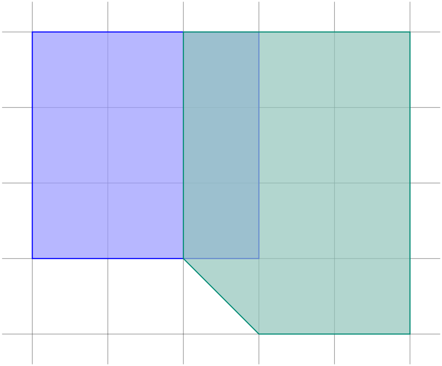

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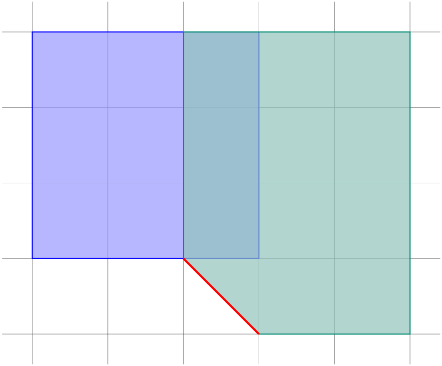

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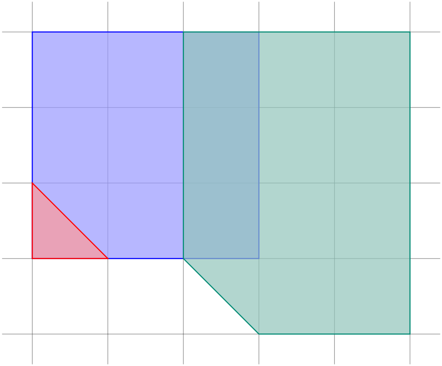

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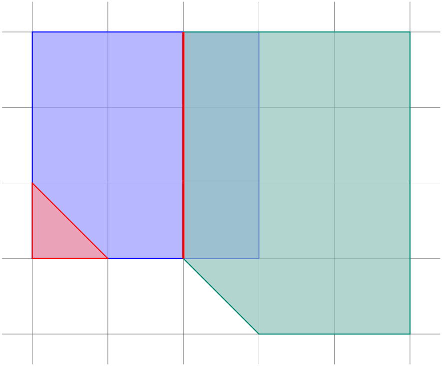

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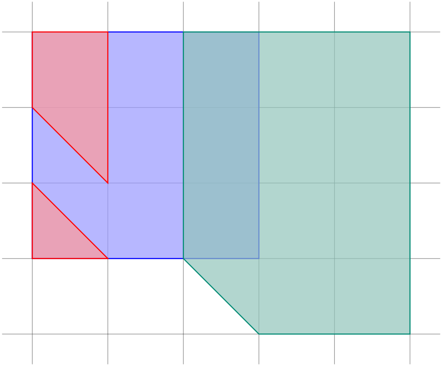

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$$\min \{ (x, y) \to (z) : z \geq 0 \land x + z \geq 0 \land y + z \geq 0 \}$$

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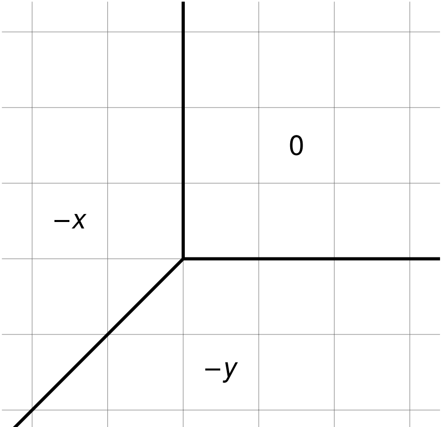

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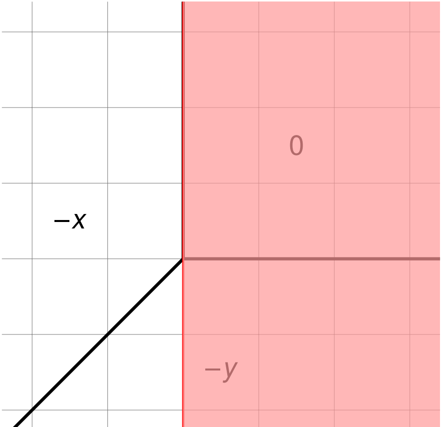

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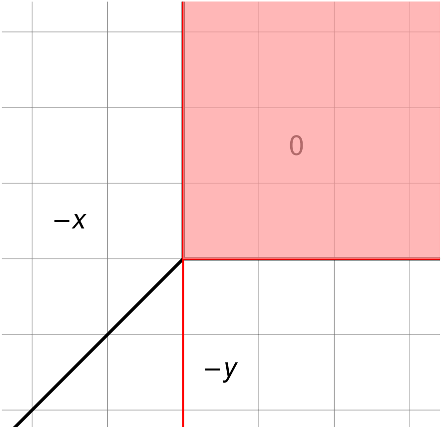

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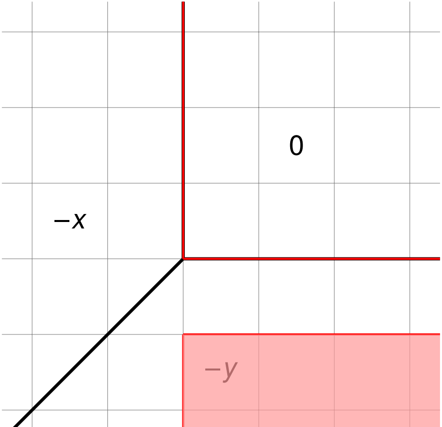

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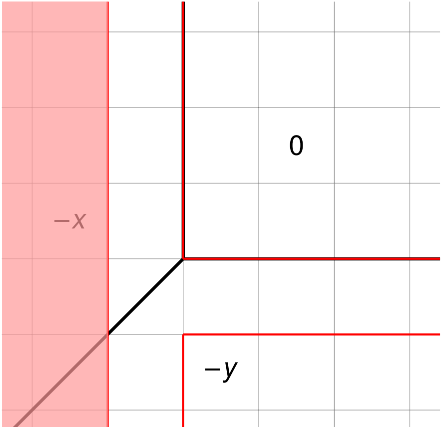

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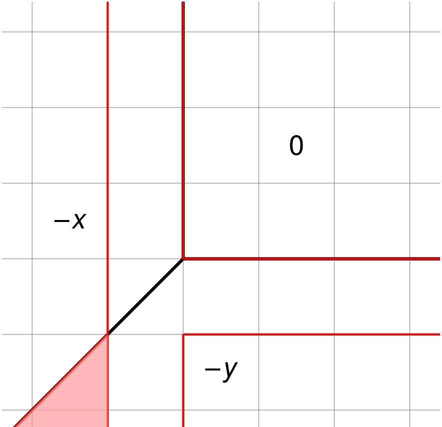

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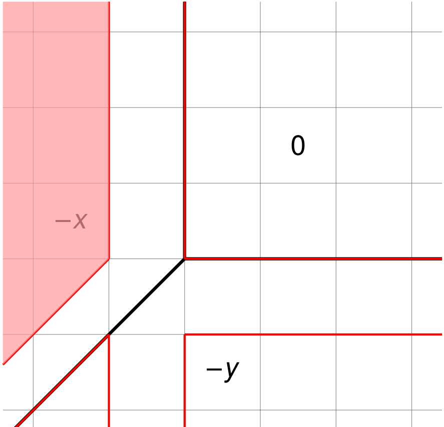

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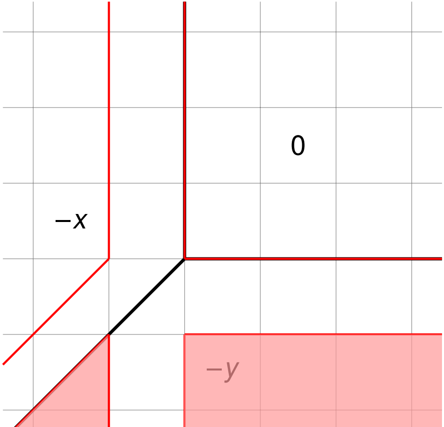

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Several operations on integer sets may introduce coalescing opportunities

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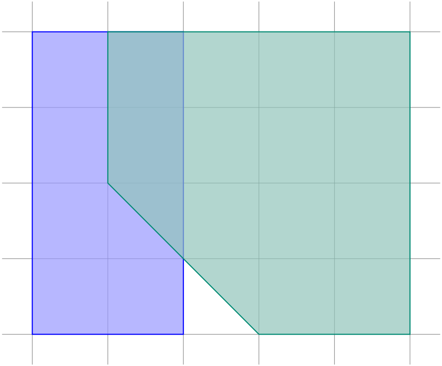

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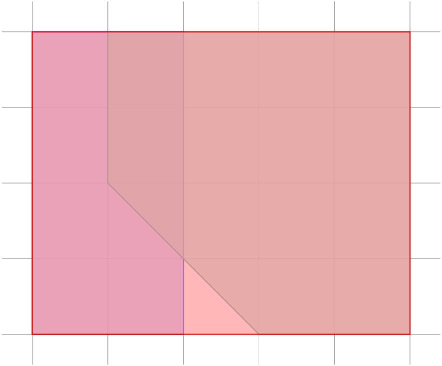

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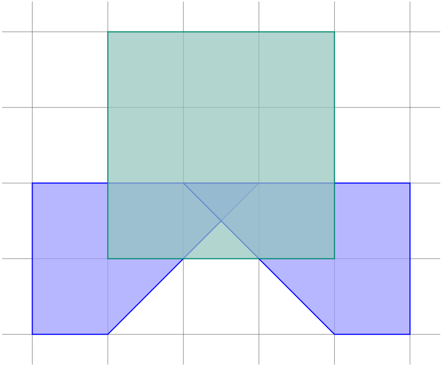

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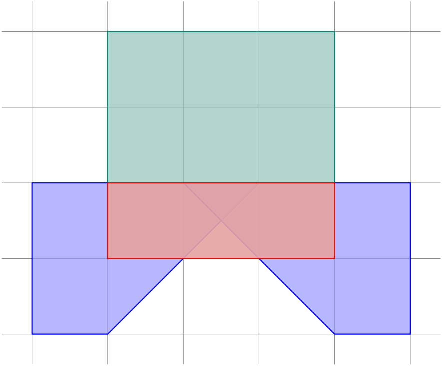
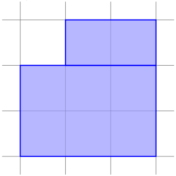


Traditional "Coalescing"

Traditional method (e.g., in CLooG with original PolyLib backend)

1. Compute convex hull H of S
2. Remove integer elements not in S from H
   $\Rightarrow H \setminus (H \setminus S)$



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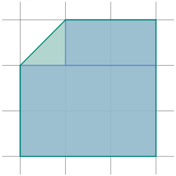

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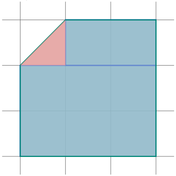

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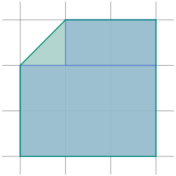

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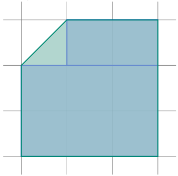

Issues:

- Convex hull may have exponential number of constraints
  We may be able to remove some of them, but we still need to compute them first.
- Constraints of convex hull may have very large coefficients
- Convex hull is an operation on *rational* sets
  - ⇒ mixture of operation on rational sets (convex hull) and integer sets (set subtraction)
  - ⇒ in `isl`, convex hull operation not fully defined on sets with existentially quantified variables
- Convex hull is costly to compute

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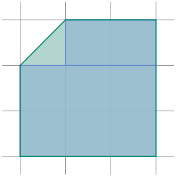

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Effect on AST Generation — `covariance`

With `isl` coalescing (in this case same result as no coalescing)

```
for (long c1 = n >= 1 ? ((n - 1) % 32) - n - 31 : 0;
     c1 <= (n >= 1 ? n - 1 : 0); c1 += 32) {
  /* .. */
}
```

With convex hull based "coalescing"

```
for (long c1 = 32 * floord(-1073741839 * n -
    32749125633, 68719476720) - 1073741792; c1 <=
    floord(715827882 * n + 357913941, 1431655765) +
    1073741823; c1 += 32) {
  /* .. */
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```

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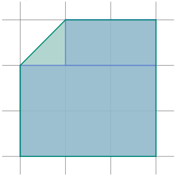

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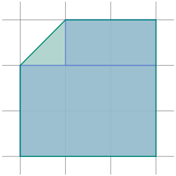

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AST Generation Times

Generation times on `isl` AST generation test cases

`isl` coalescing	16.0s
no coalescing	16.3s
convex hull (FM)	24m00s
convex hull (wrapping)	6m40s

Note: `isl` may not have the most efficient convex hull implementation

However, double description based implementations are costly too

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- Constraints adjacent to inequality
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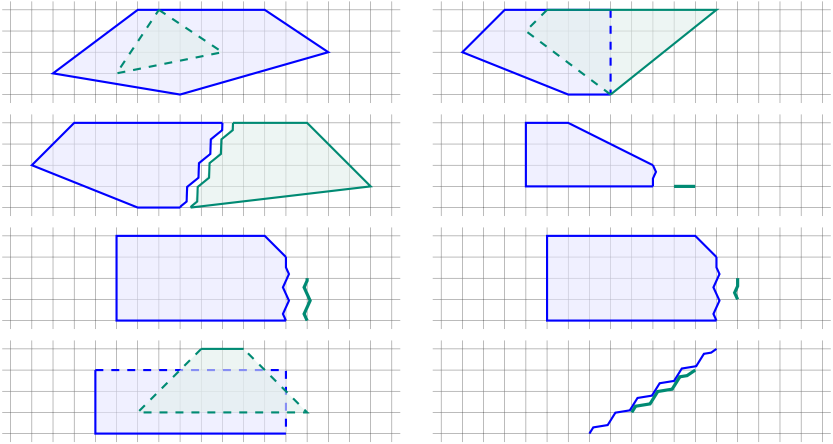
- Wrapping
- Existentially Quantified Variables

3 Conclusions

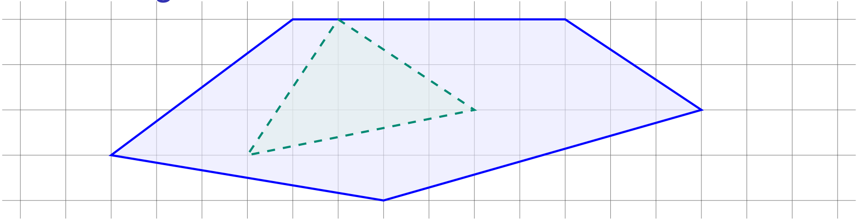
Coalescing in `isl`

Coalescing in `isl`

- never increases the total number of constraints
- based on solving LP problems with same dimension as input set
- recognizes a set of patterns



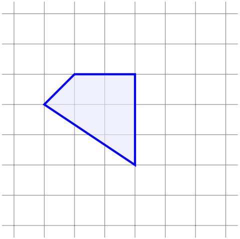
Coalescing Cases



Constraint types

Given two disjuncts A and B

For each affine constraint $t(\mathbf{x}) \geq 0$ of A, determine its effect on B



Note: affine expression $t(\mathbf{x}) \geq 0$ has integer coefficients
min and max computed using (incremental) LP solver

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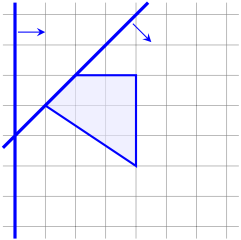
  $\Rightarrow$valid constraint



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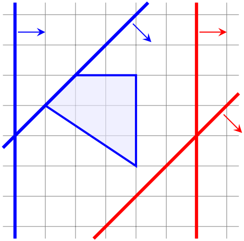
- $\max t(\mathbf{x}) < 0$ over B
  $\Rightarrow$separating constraint



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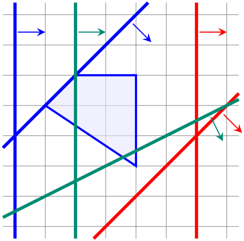

- otherwise (attains both positive and negative values over B)
  $\Rightarrow$cut constraint

Note: affine expression $t(\mathbf{x}) \geq 0$ has integer coefficients
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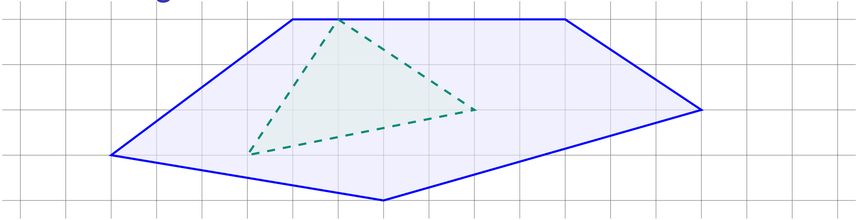
Coalescing Cases



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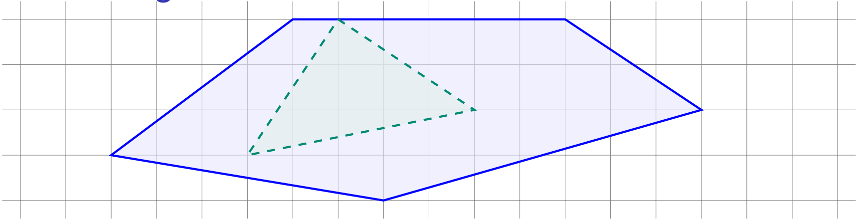

1. All constraints of A are valid for B

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Constraint $t(\mathbf{x}) \geq 0$

- valid: $\min t(\mathbf{x}) > -1$
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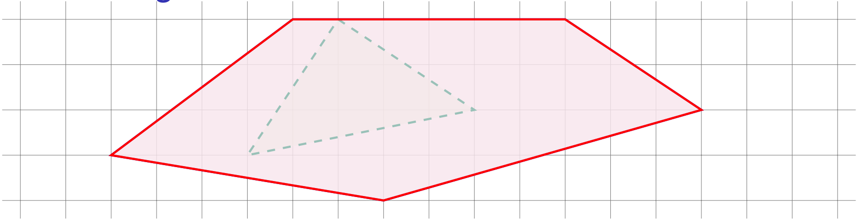
Coalescing Cases



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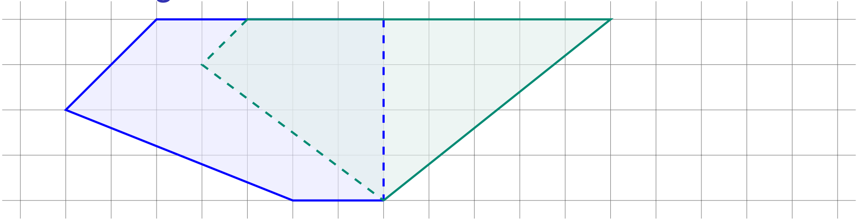

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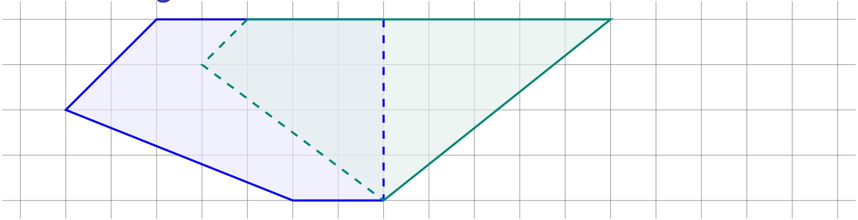

1. All constraints of A are valid for B

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2. Neither A nor B have separating constraints and all cut constraints of A are valid for the cut facets of B

   $\Rightarrow$ replace $A \cup B$ by set bounded by all valid constraints

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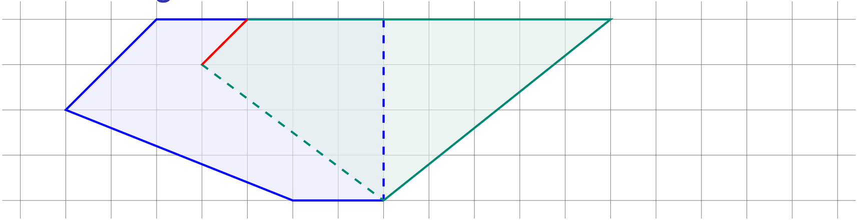

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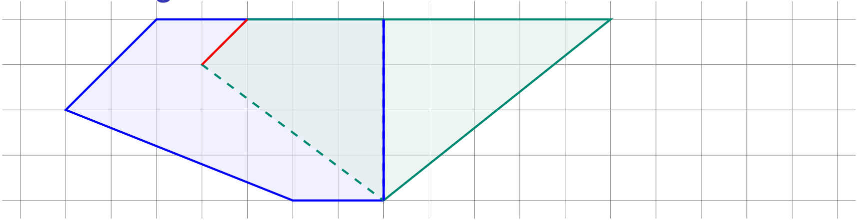

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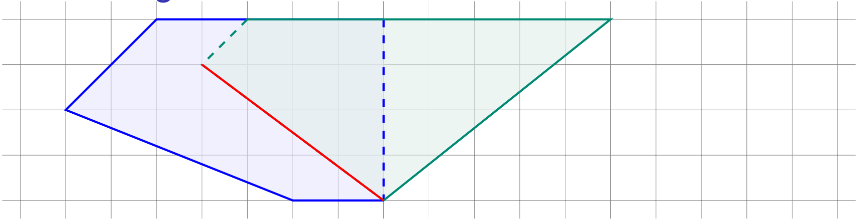

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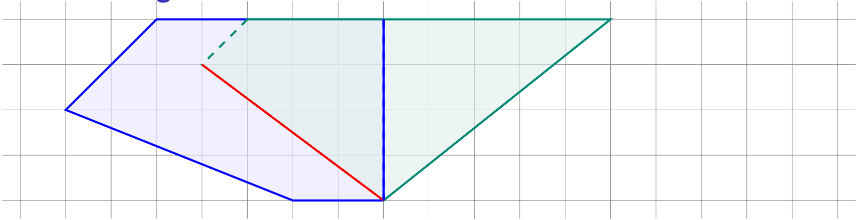

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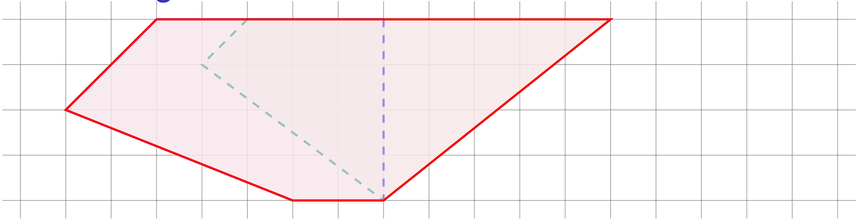

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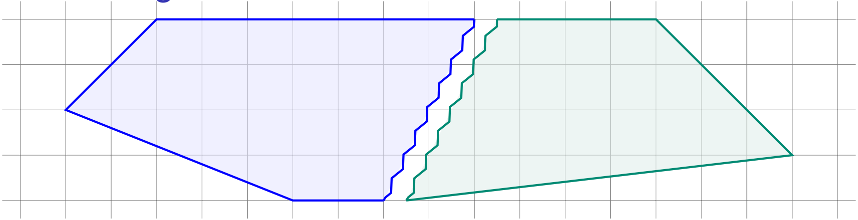

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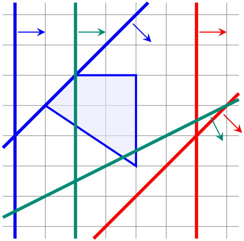
Constraint types

Given two disjuncts A and B

For each affine constraint $t(\mathbf{x}) \geq 0$ of A, determine its effect on B



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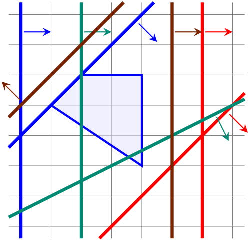

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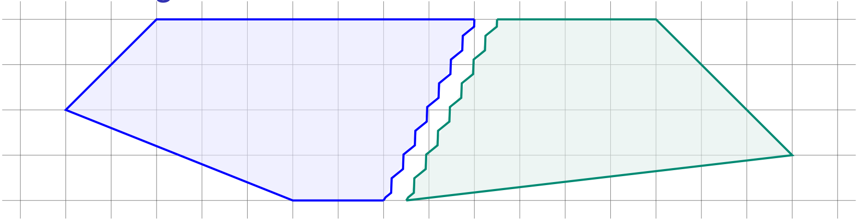

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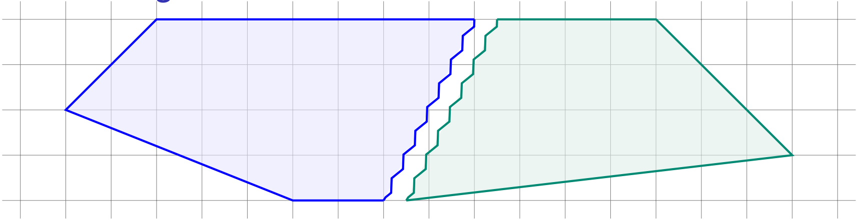

3. single pair of adjacent inequalities (other constraints valid)

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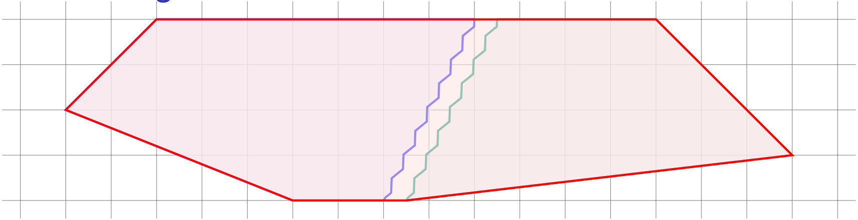
Coalescing Cases



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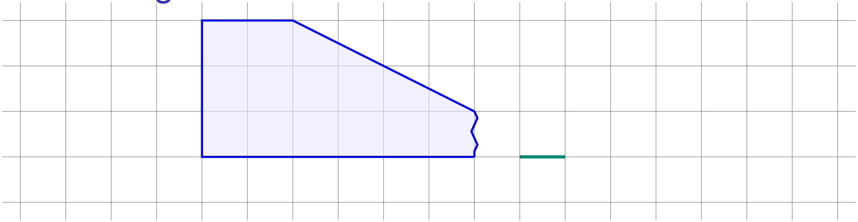

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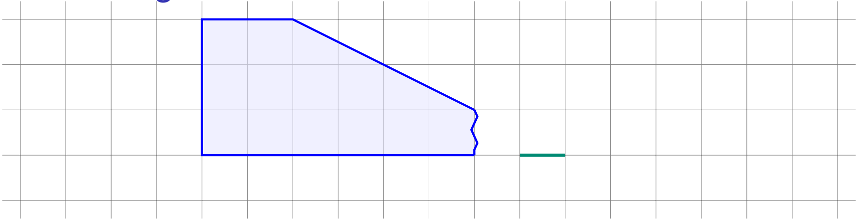

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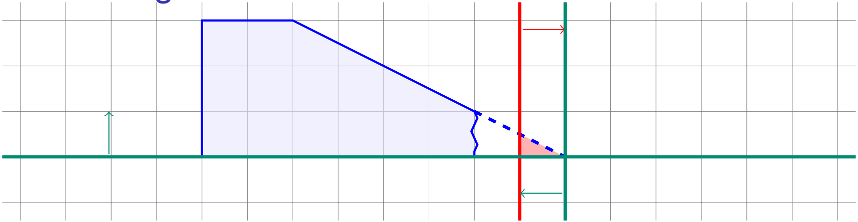

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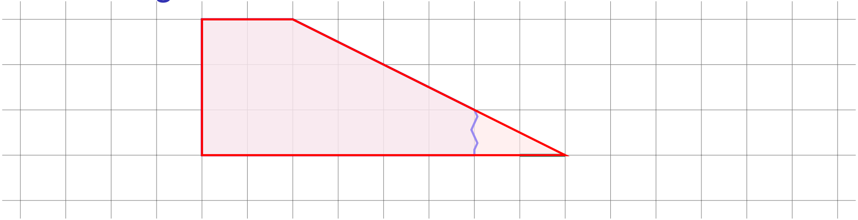

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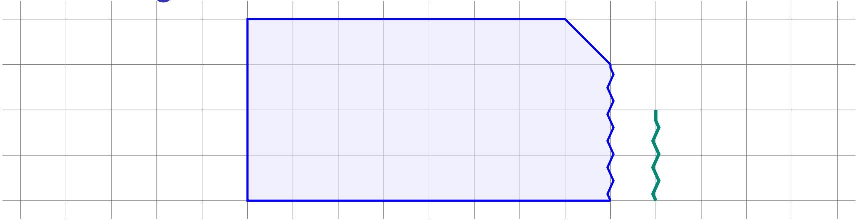

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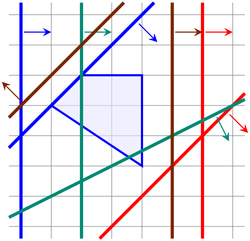

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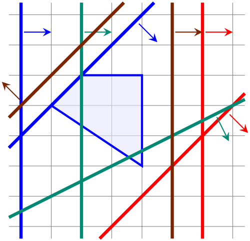

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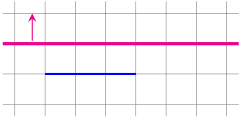

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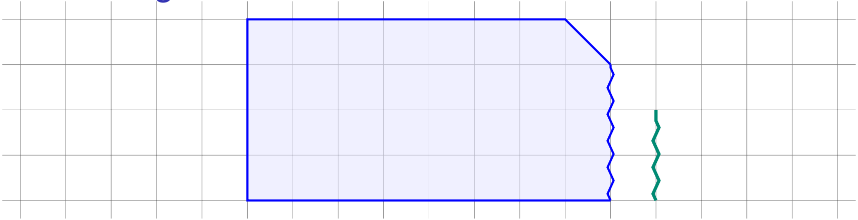

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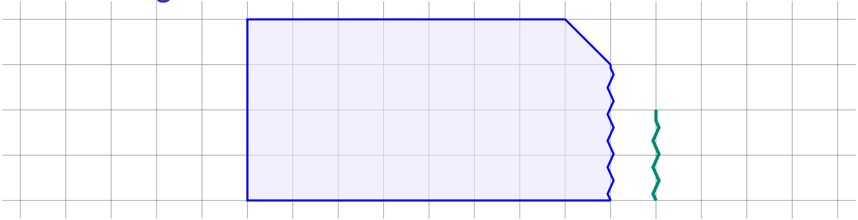

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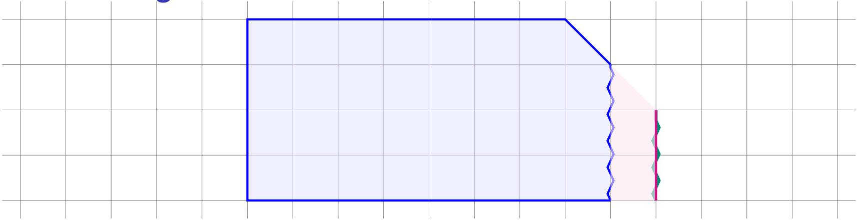

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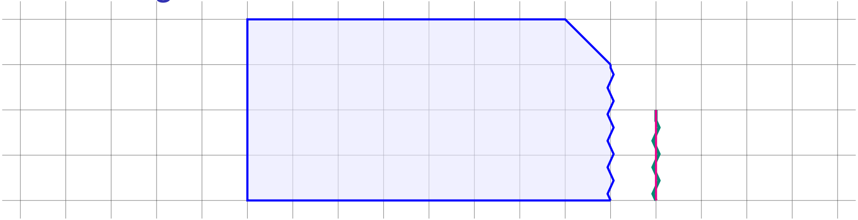

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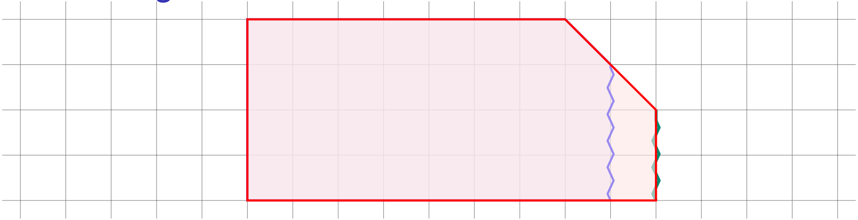

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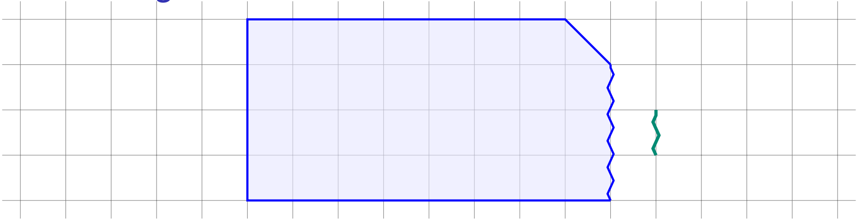

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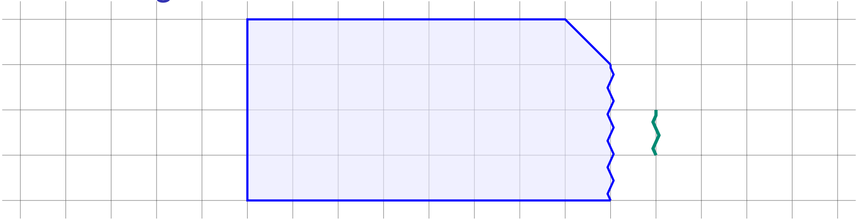

6. A has single inequality adjacent to equality of B (other constraints of A are valid)
   Non-valid constraints of B (except $t(\mathbf{x}) \leq -1$) can be wrapped around $t(\mathbf{x}) \geq -1$ to include A

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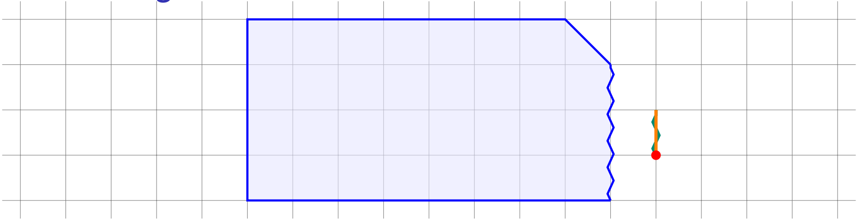

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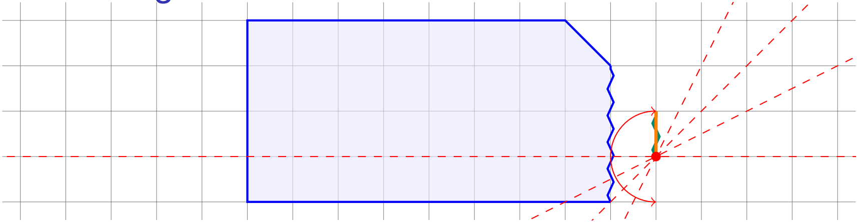

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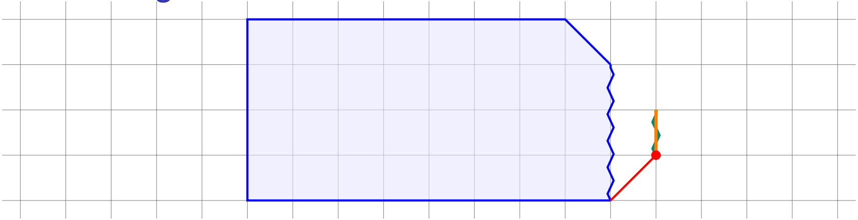

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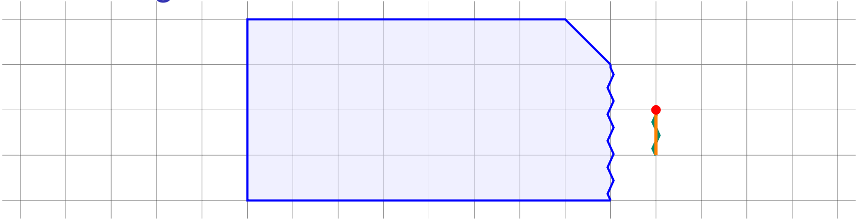

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- valid: $\min t(\mathbf{x}) > -1$
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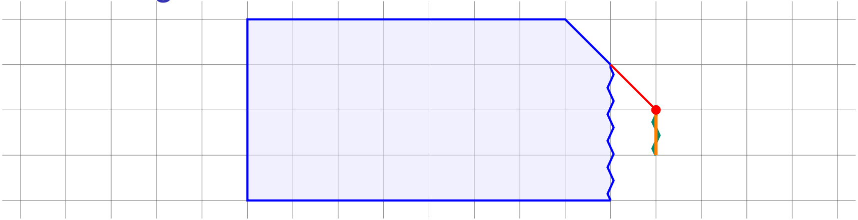
Coalescing Cases



6. *A* has single inequality adjacent to equality of *B* (other constraints of *A* are valid)
   Non-valid constraints of *B* (except $t(\mathbf{x}) \leq -1$) can be wrapped around $t(\mathbf{x}) \geq -1$ to include *A*
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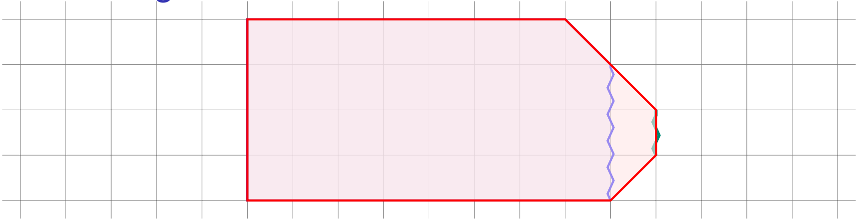

6. A has single inequality adjacent to equality of B (other constraints of A are valid)
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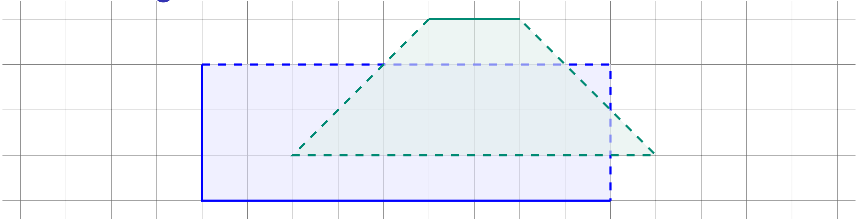

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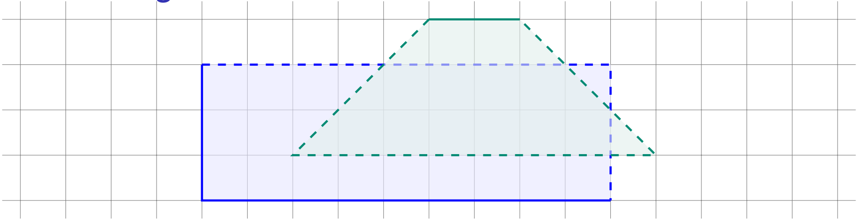

7. *B* extends beyond *A* by at most one and all cut constraints of *B* can be wrapped around shifted facet of *A* to include *A*

   $\Rightarrow$ replace $A \cup B$ by set bounded by all valid constraints and all wrapped constraints
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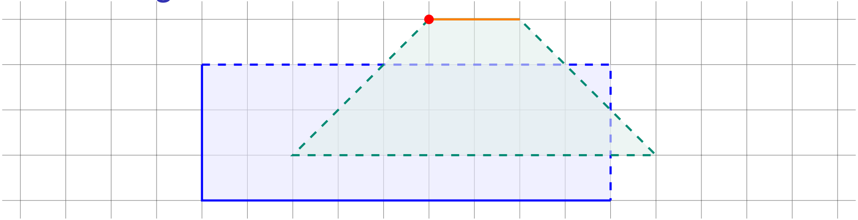

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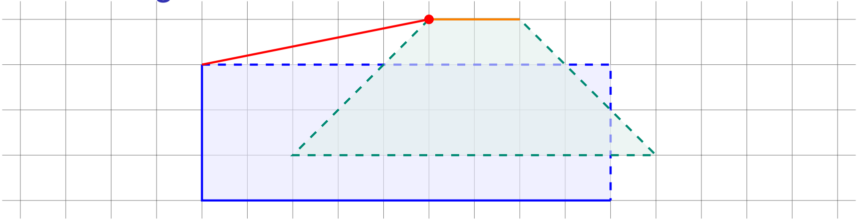

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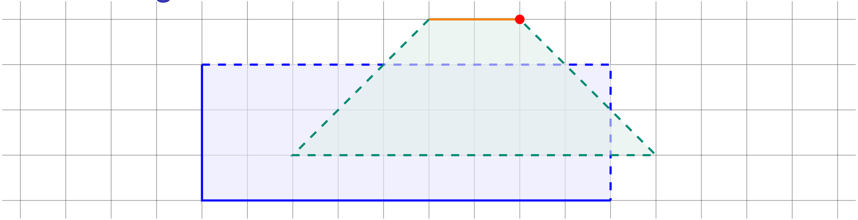

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Constraint $t(\mathbf{x}) \geq 0$

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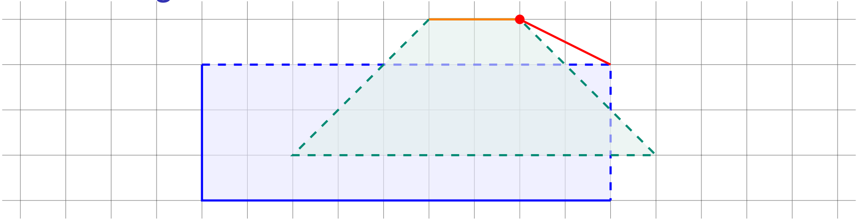

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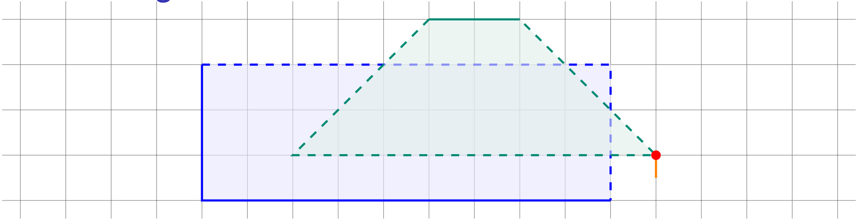

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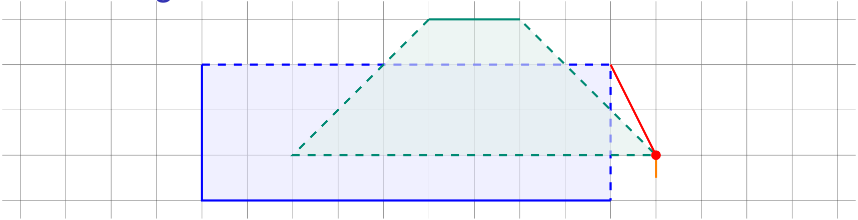

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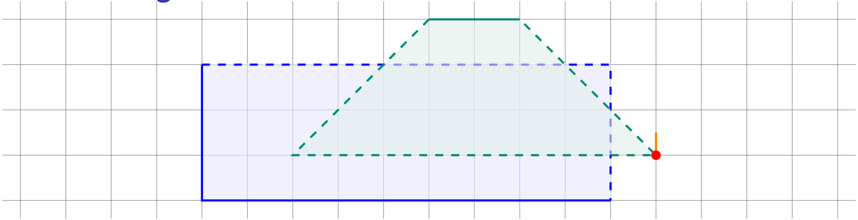

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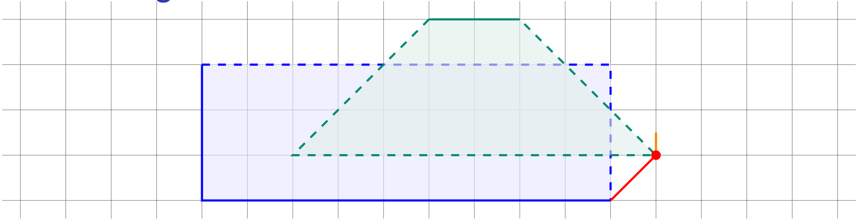

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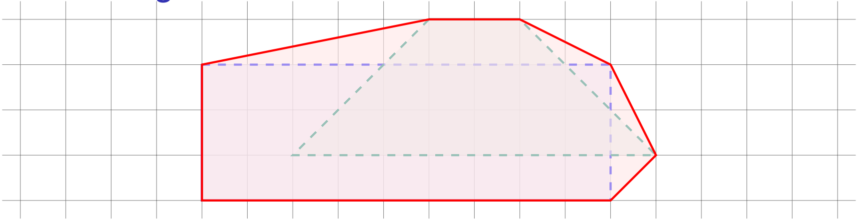

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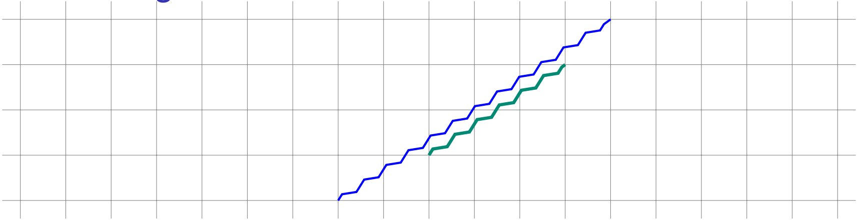

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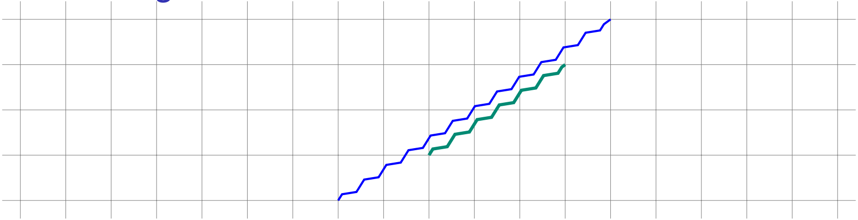

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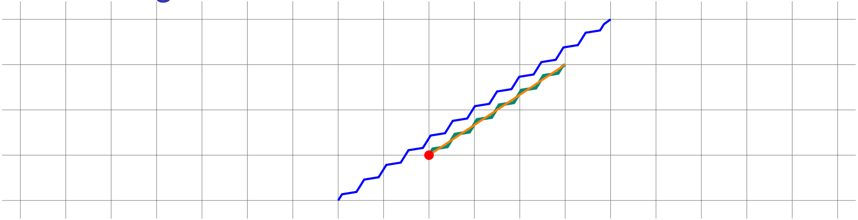

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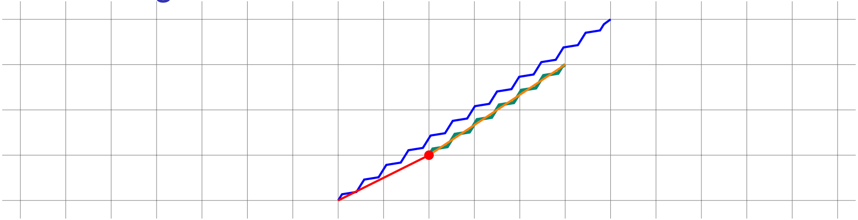

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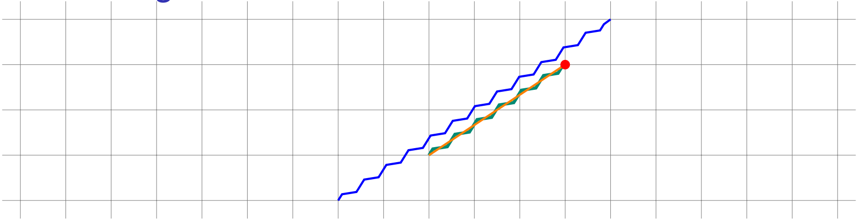

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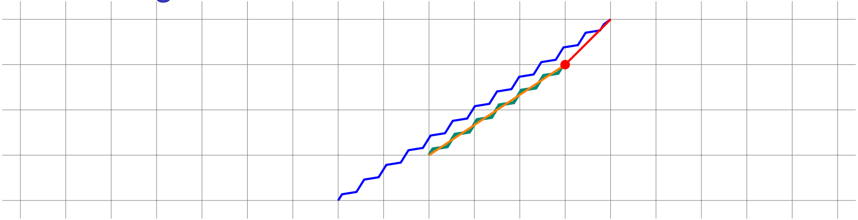

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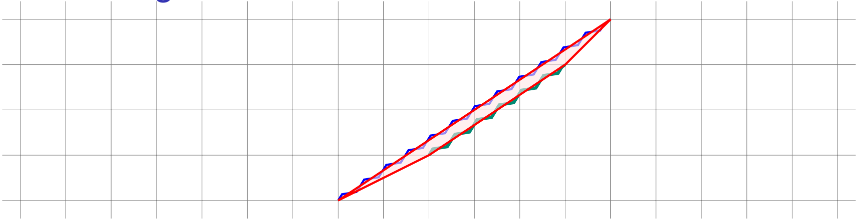

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Existentially Quantified Variables and Equalities

- Quantifier elimination in `isl` replaces existentially quantified variables by integer divisions of affine expressions in other variables
- These integer divisions are sorted prior to coalescing

- *A* and *B* have same number of integer divisions/existentials
  ⇒ try all cases
- integer divisions of *A* form subset of those of *B*
  (after exploiting equalities of *B*)
  ⇒ check if *B* is a subset of *A*
- integer divisions of *B* form subset of those of *A* and
  equalities of *B* simplify away the integer divisions of *A* not in *B*
  ⇒ introduce integer divisions in *B* and try all cases

Outline

1 Introduction and Motivation
- Polyhedal Model
- The need for coalescing
- Traditional “Coalescing”

2 Coalescing in `isl`
- Rational Cases
- Constraints adjacent to inequality
- Constraints adjacent to equality
- Wrapping
- Existentially Quantified Variables

3 Conclusions

Conclusions

- it is important to keep the number of disjuncts in a set representation as low as (reasonably) possible
- coalescing in `isl`
  - never increases the total number of constraints
  - based on solving LP problems with same dimension as the original set
  - recognizes a set of patterns